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ON A REPRESENTATION OF SOLUTIONS OF THE ANTIPLANE DEFORMATION PROBLEM OF A PHYSICALLY NONLINEAR ELASTIC MEDIUM

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A general representation of solutions in the problem of antiplane deformed state of the medium with governing relations of the type met in the deformational theory is given. It is shown that by imposing very weak restrictions on the form of the uniaxial stress-strain diagram, the basic equations of the problem can be reduced to a generalized Cauchy–Riemann system defining, as the representation, generalized analytic functions of the complex variable. Thus the problem of antiplane deformation becomes a boundary value problem of the theory of p- analytic function [1]. A problem of a crack emerging at the boundary of the half-plane is considered, existence and uniqueness of the solution is proved and the asymptotics of the solution for arbitrary values of the loading parameter is obtained. The possibility of representing the solutions in closed form is considered.

The above problem was investigated in [2] under more stringent assumptions concerning the stress-strain relations.

1. The basic system of equations of the problem has the form

$$\frac{\partial\tau_x}{\partial x}+\frac{\partial\tau_y}{\partial y}=0,\quad \frac{\partial\gamma_x}{\partial y}-\frac{\partial\gamma_y}{\partial x}=0 \tag{1.1}$$
$$\tau_x=\frac{\tau(\gamma)}{\gamma}\gamma_x,\quad \tau_y=\frac{\tau(\gamma)}{\gamma}\gamma_y$$

Here τ_x, τ_y, $^1/_2\,\gamma_x$, $^1/_2\,\gamma_y$ are the shear components of the stress and strain tensors in the Cartesian coordinate system (the index z is omitted for simplicity), while τ and $^1/_2\,\gamma$ denote the intensities of their deviators. It is assumed that the process of deformation is quasi-static.

We shall show that certain problems of the steady distribution of current in nonlinearly conducting media (see e.g. [3]) also lead to a system of the form (1.1) (this is apparently the first time that such an analogy has been observed).

The known representations of the solutions of (1.1) refer to various type approximations of the relation $\tau(\gamma)$. These representations made it possible to obtain solutions of the problems of concentration of stresses and deformations in elastoplastic media [4], important for the practical applications. We shall mention, in particular, [2] where a representation for (1.1) was obtained for the case of a semi-plane with an angular cutout under the assumption that the initial segment of the diagram was linear.

Let us consider the system (1.1) in the general case, assuming that the function $\tau(\gamma)\in C_1\,(0,\infty)$. Introducing the governing equations into the equation of equilibrium, we find from (1.1)

$$\frac{\partial\gamma_x}{\partial x}\left(\frac{\tau}{\gamma}+\gamma_x{}^2\frac{\tau'\gamma-\tau}{\gamma^3}\right)+\frac{\partial\gamma_y}{\partial y}\left(\frac{\tau}{\gamma}+\gamma_y{}^2\frac{\tau'\gamma-\tau}{\gamma^3}\right)+ \qquad (1.2)$$
$$2\gamma_x\gamma_y\,\frac{\tau'\gamma-\tau}{\gamma^3}\frac{\partial\gamma_x}{\partial y}=0,\quad \frac{\partial\gamma_y}{\partial x}-\frac{\partial\gamma_x}{\partial y}=0$$

The system (1.2) is quasi-linear and of first order in $\gamma_x(x, y)$, $\gamma_y(x, y)$. It can be shown using the well known methods that (1.2) is elliptical when $\tau'(\gamma)>0$, hyperbolic when $\tau'(\gamma)<0$ and parabolic when $\tau'(\gamma)=0$. For the elastoplastic problems the above cases correspond to a hardening, softening [5] and a perfectly plastic medium, and for the media with nonlinear Ohm's law, to the stable, unstable and a saturation segment respectively [3].

Taking into account the reducibility [6] of the system, we linearize it with the help of a hodograph transformation with the independent variables γ_y, γ_x, so that

$$\frac{\partial x}{\partial\gamma_y}=\frac{1}{\Delta}\frac{\partial\gamma_x}{\partial y},\quad \frac{\partial x}{\partial\gamma_x}=-\frac{1}{\Delta}\frac{\partial\gamma_y}{\partial y},\quad \frac{\partial y}{\partial\gamma_y}=-\frac{1}{\Delta}\frac{\partial\gamma_x}{\partial x} \qquad (1.3)$$
$$\frac{\partial y}{\partial\gamma_x}=\frac{1}{\Delta}\frac{\partial\gamma_y}{\partial x},\quad \Delta=\frac{\partial\gamma_y}{\partial x}\frac{\partial\gamma_x}{\partial y}-\frac{\partial\gamma_y}{\partial y}\frac{\partial\gamma_x}{\partial x}$$

Let us inspect the case when the transformation is degenerate. Satisfying the equation $\Delta=0$, we set

$$\frac{\partial\gamma_x}{\partial x}=\lambda\frac{\partial\gamma_x}{\partial y},\quad \frac{\partial\gamma_y}{\partial x}=\lambda\frac{\partial\gamma_y}{\partial y}\quad (\lambda=\lambda(x, y)) \qquad (1.4)$$

Differentiating the second equation of (1.2) with respect to x and using relations (1.4), we obtain the following equation for $\lambda(x, y)$:

$$\frac{\partial\lambda}{\partial x}-\lambda\frac{\partial\lambda}{\partial y}=0$$

The general integral of this equation has the form $\lambda x+y=C(\lambda)$, therefore in the degenerate case the system (1.2) has rectilinear characteristics belonging to the family $\lambda=\mathrm{const}$.

Using the relations (1.4) we can confirm that the deformed state is simple [6] ($\gamma_x=\gamma_x(\lambda x+y)$, $\gamma_y=\gamma_y(\lambda x+y)$). The latter is realized e.g. near a circular, stress-free opening in an elastic, perfectly plastic medium.

Returning to the general case $\Delta\neq 0$ and inserting (1.3) into (1.2), we obtain

$$\frac{\partial y}{\partial\gamma_y}\left(\frac{\tau}{\gamma}+\gamma_x{}^2\,\frac{\tau'\gamma-\tau}{\gamma^3}\right)+\frac{\partial x}{\partial\gamma_x}\left(\frac{\tau}{\gamma}+\gamma_y{}^2\frac{\tau'\gamma-\tau}{\gamma^3}\right)-2\gamma_x\gamma_y\frac{\partial x}{\partial\gamma_y}=0 \qquad (1.5)$$
$$\frac{\partial x}{\partial\gamma_y}-\frac{\partial y}{\partial\gamma_x}=0$$

Equations analogous to (1.5) were obtained in a different manner in [2].

Let us transform the system (1.5) using the logarithmic polar coordinate system

$$\frac{\partial x}{\partial\rho}\sin\varphi+\frac{\partial x}{\partial\varphi}\varkappa\cos\varphi+\frac{\partial y}{\partial\rho}\cos\varphi-\frac{\partial y}{\partial\varphi}\varkappa\sin\varphi=0 \qquad (1.6)$$

$$\frac{\partial x}{\partial \rho}\cos\varphi - \frac{\partial x}{\partial \varphi}\sin\varphi - \frac{\partial y}{\partial \rho}\sin\varphi - \frac{\partial y}{\partial \varphi}\cos\varphi = 0$$
$$(\gamma_y = \gamma\cos\varphi,\ \gamma_x = \gamma\sin\varphi,\ \rho = \ln\gamma,\ \varkappa = \tau'\gamma / \tau)$$

Introducing the unknown functions $a(\rho, \varphi) = x\sin\varphi + y\cos\varphi$, $b(\rho, \varphi) = x\cos\varphi - y\sin\varphi$ and a system of independent variables $\xi = \varphi$

$$\eta = \int \sqrt{\varkappa}\, d\rho$$

we obtain from (1.6)

$$\frac{\partial a}{\partial \xi} - \sqrt{\varkappa}\frac{\partial b}{\partial \eta} - b = 0, \quad \frac{\partial a}{\partial \eta} + \sqrt{\varkappa}\frac{\partial b}{\partial \xi} + \sqrt{\varkappa} a = 0 \quad (\varkappa \geqslant 0)$$

The above system admits a transformation of the dependent variables, and the transformation reduces it to a generalized Cauchy – Riemann system. Setting

$$a = \alpha\exp\left(-\int p\, d\eta\right), \quad b = \beta\exp\left(-\int\frac{d\eta}{p}\right) \tag{1.7}$$
$$P = \frac{1}{p}\exp\left(\int\left(\frac{1}{p} - p\right)d\eta\right), \quad p = \sqrt{\varkappa}$$

we obtain

$$\frac{\partial \alpha}{\partial \xi} - \frac{1}{P}\frac{\partial \beta}{\partial \eta} = 0, \quad \frac{\partial \alpha}{\partial \eta} + \frac{1}{P}\frac{\partial \beta}{\partial \xi} = 0 \tag{1.8}$$

Thus $f(\zeta) = \alpha + i\beta$ is a generalized analytic function of the variable $\zeta = \xi + i\eta$ with the characteristic P. The latter depends on the variable η only, therefore $f(\zeta)$ belongs to an invariant class [1].

It follows therefore that the problem of nonlinear – elastic deformation under the conditions of antiplane state, reduces to a boundary value problem of the theory of P–analytic functions (pseudoanalytic) developed in [1, 7], et al.

The known applications of this theory to the problems of mechanics refer chiefly to the theory of filtration, to the axisymmetric problem of the theory of elasticity and to the plane problem of the theory of plasticity where it was found helpful in formulating the majorant methods and gave, in a number of cases, closed solutions [1].

2. The problem formulated above is linear (e. g. it represents the generalized Riemann – Hilbert conjugation problem, provided that the boundary conditions given on the known mappings of the region boundaries in the z - plane onto the ζ - plane are linear. In particular, this always takes place in the case of the boundary value problems for polygons with homogeneous boundary conditions.

As an example, we shall use the representation obtained in Sect. 1 for a semiplane $D = \{x \geqslant 0,\ |y| < \infty\}$ with a crack or cut $L = \{0 \leqslant x \leqslant l,\ y = 0\}$, which is subjected to a homogeneous shear of magnitude γ_∞ at infinity.

Let us introduce the complex function

$$\Omega(\zeta) = ((\exp\zeta)^2 - 1)^{1/2} \quad (\eta = \operatorname{Im}\zeta, \quad \eta(\rho_\infty) = 0, \quad \rho_\infty = \ln\gamma_\infty)$$

which maps, together with the hodograph transformation, $\zeta = \zeta(z)$, the region D onto the semiplane $D_+^\circ = \{\operatorname{Re}\Omega \geqslant 0\}$. Since $\Omega(\zeta)$ is analytic in D, the system (1.8) becomes

$$\frac{\partial\beta}{\partial\Omega_1} - \frac{1}{R}\frac{\partial\alpha}{\partial\Omega_2} = 0, \quad \frac{\partial\beta}{\partial\Omega_2} + \frac{1}{R}\frac{\partial\alpha}{\partial\Omega_1} = 0$$
$$R = 1 / P\,(\ln|\sqrt{\Omega^2+1}|), \quad \Omega = \Omega_1 + i\,\Omega_2 \tag{2.1}$$

and defines the function $f(\Omega) = \beta + i\alpha$, R-analytic in D_+°.

Remembering that $\xi = 0$ when $x = 0$ and $\xi = \pm\pi/2$ when $y = 0, 0 \leqslant x \leqslant l$, we obtain, in accordance with (1.7), the boundary condition

$$\beta = \mathrm{Re} f = 0 \quad (\Omega_1 = 0) \tag{2.2}$$

Let us, in addition, define the solution in the semiplane $D_-^\circ = \{\Omega_1 < 0\}$ according to the rule $f(\Omega) = -f(-\overline{\Omega})$. We see from (2.2) that the function $f(\Omega)$ is continuous on the imaginary axis and satisfies the relations (2.1) in $D^\circ = D_+^\circ \cup D_-^\circ$, except at the single-valued singularities $\Omega = 0, \infty$. In order to find the asymptotics at the above points, we note that according to (1.7)

$$\alpha = a\tau, \quad \beta = b\gamma, \quad f(\Omega) = b\gamma + ia\tau \tag{2.3}$$

so that when $z \to l$ and $z \to \infty$, we obtain from (2.3), respectively,

$$f \sim l\,(\gamma\cos\xi + i\tau\sin\xi), \quad f \sim b\gamma_\infty + i\tau_\infty a \quad (b + ia = ze^{i\xi}) \tag{2.4}$$

Satisfying the first asymptotic of (2.4), we set

$$f(\Omega) = f_1(\Omega) + F(\Omega) \tag{2.5}$$
$$f_1(\Omega) = l\cos\xi\exp\left(\int_0^\eta \frac{d\eta}{p}\right) + il\sin\xi\exp\left(\int_0^\eta p\,d\eta\right)$$
$$(\zeta = \ln\sqrt{\Omega^2+1} \to \infty)$$

It can easily be shown that the function $f_1(\Omega)$ is R-analytic and has, according to (2.5), a first order pole at $\Omega = \infty$. It follows therefore that $F(\Omega)$ is holomorphic in the neighborhood of $\Omega = \infty$ in the sense of generalized analytic functions.

When $z \to \infty$ and $\Omega \to 0$, (2.1) yields the asymptotics

$$\beta + i\frac{\alpha}{R(0)} = \frac{c}{\Omega^n} + o\left(\frac{1}{\Omega^n}\right), \quad n = 1, 2, \ldots \tag{2.6}$$

When $R(0) = 1$, the solution should coincide with the linearly elastic solution, therefore, remembering that the spectrum of the parameter n is discrete, we must set $n = 1$ in (2.6). In this case the function $F(\Omega)$ has a first order pole at $\Omega = 0$ and can be written in the form

$$F(\Omega) = f_2(\Omega) + f_3(\Omega) \tag{2.7}$$

where the singularity $f_3(\Omega)$ is, in accordance with (2.6), removable.

We note that $f_1(\Omega)$ and $f_2(\Omega)$ are holomorphic (or have removable singularities) at $\Omega = \infty$ and 0 respectively, while the R-analytic function $f_3(\Omega)$ has no singularities in D° except, perhaps, some removable ones.

The pseudoanalytic function can be determined by specifying a set of its singularities in the complete plane, consequently from (2.5)-(2.7) we can assert that a solution of the problem exists.

To prove uniqueness it is sufficient to assume that another solution $g(\Omega)$ exists and apply the generalized Liouville theorem [1] to the function $f(\Omega) - g(\Omega)$ R - analytic in D°.

3. Let us now consider whether the linear boundary value problems for the system (1.8) have solutions in closed form. In [1] it was shown that the sufficient condition for these solutions to exist is, that its characteristic can be written in the form

$$P(\eta) = c\eta^{k}, \quad k \geqslant 0 \tag{3.1}$$

Let us find the relations $\tau = \tau(\gamma)$ which satisfy the condition (3.1). Assuming that in the last relation of (1.7)

$$\exp\left(\int\left(\frac{1}{p} - p\right) d\eta\right) = p\eta^{k}$$

we find that the function $p(\eta)$ satisfies the equation

$$p'\eta + kp + \eta p^{2} = \eta \tag{3.2}$$

which defines it to within the constants k and $p(0)$. This implies that the corresponding class of the diagrams $\tau(\gamma)$ is three - parametric. In particular, for a medium satisfying Hooke's Law we must set $k = 0$, $p(0) = 1$.

A number of concrete problems can be solved using the solutions of (3.2) to approximate segments of the prescribed relationship and making use of the representations for the η^{k}-analytic functions. Another way of constructing the solutions consists of writing the general integral of the system in the form of linear combinations of analytic functions and their derivatives (some cases relating to the plane problem of the theory of plasticity were studied in [8]). Let

$$p(\eta) = 1 - \varepsilon(\eta), \ \sup|\varepsilon(\eta)| = \delta \quad \text{on} \quad \eta_0 \leqslant \eta \leqslant \eta_1.$$

The characteristic of the function at $\delta = O(1)$ has the form

$$P = 1 + 2\delta t(\eta - \eta_0) + \varepsilon(\eta) + O(\delta^{2}) \quad (0 \leqslant |t| \leqslant 1)$$

and this implies that the representation uses the linearly elastic solution and its derivatives.

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